CS103
FALL 2025

Lecture 07:
Functions

Part 2 of 2

Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

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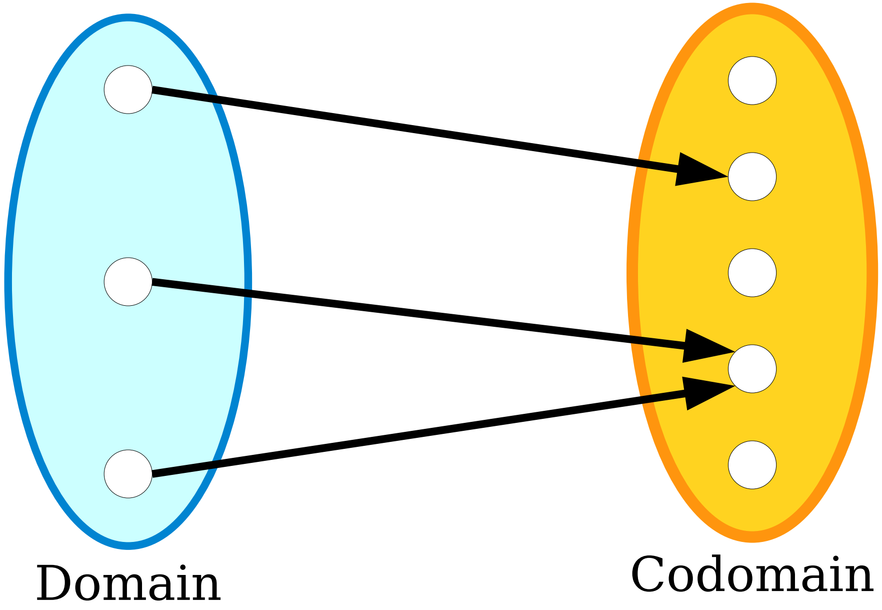
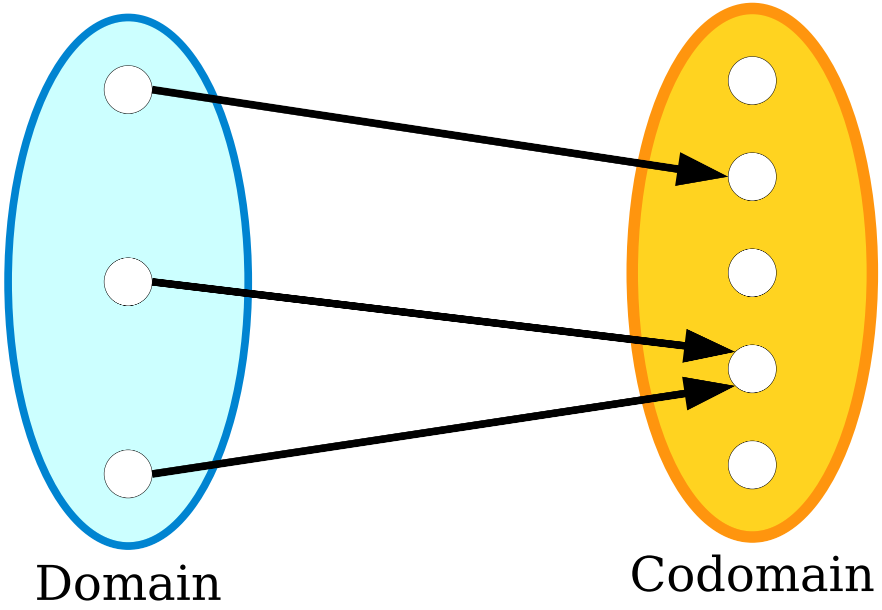


Recap from Last Time

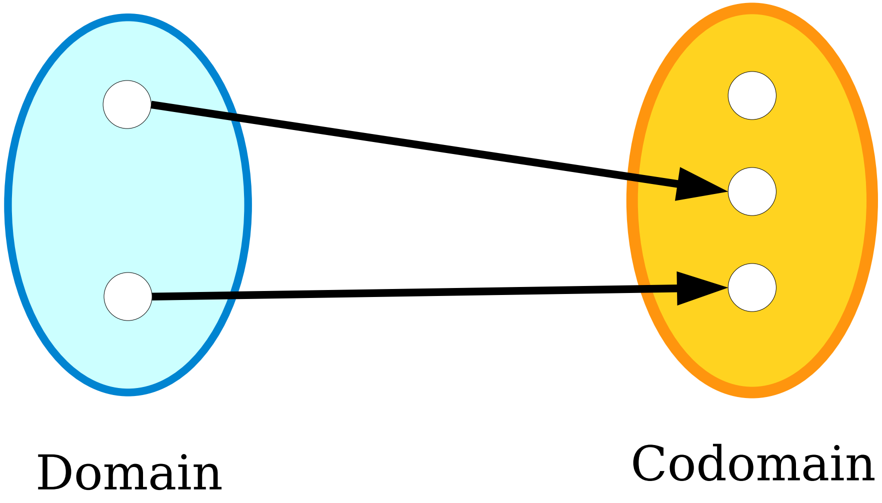


Is it a function? **Yes!**

Is it an injection? **No.**

Is it a surjection? **No.**

Recap from Last Time



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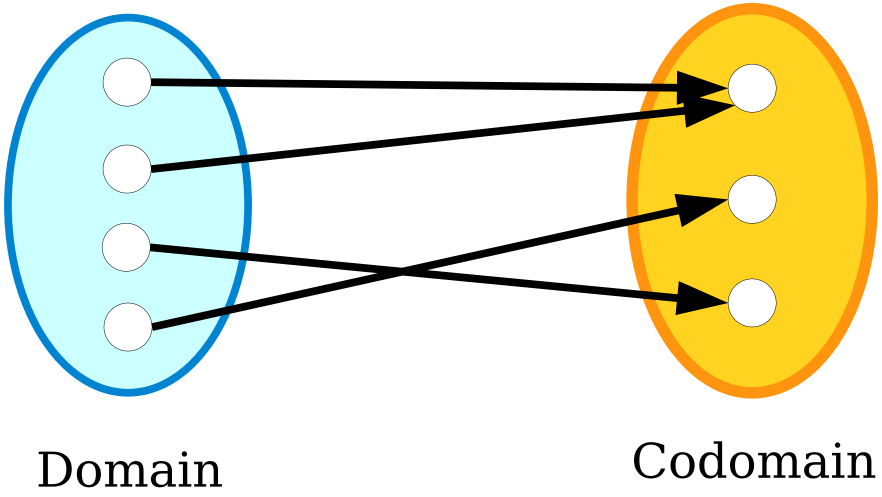


***Is it a** function?* **Yes!**

***Is it an** injection?* **No.**

***Is it a** surjection?* **Yes!**

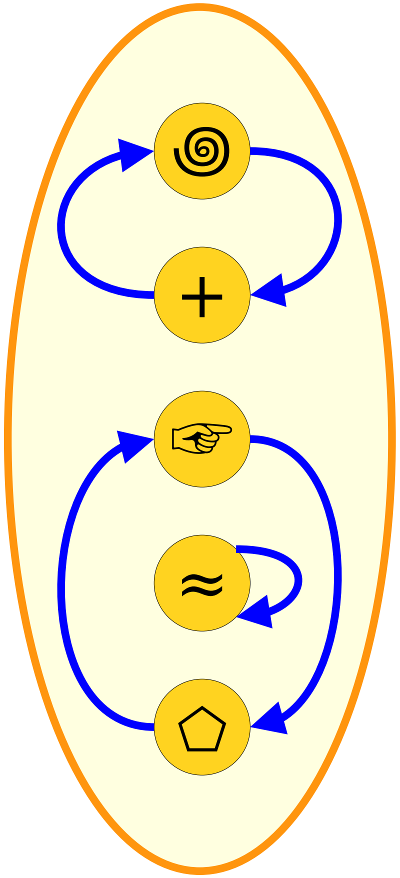
Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** when the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.



		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

New Stuff!

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

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Given the predicates

Bird(b), which says b is a bird;
Heron(h), which says h is a heron; and
Feathers(x), which says x has feathers,

translate the theorem into first-order logic.

Answer at
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$$\underbrace{(\forall b.\,(\mathit{Bird}(b) \rightarrow \mathit{Feathers}(b)))}_{\substack{\text{All birds} \\ \text{have feathers}}} \rightarrow \underbrace{(\forall h.\,(\mathit{Heron}(h) \rightarrow \mathit{Feathers}(h)))}_{\substack{\text{All herons} \\ \text{have feathers}}}$$

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Proof: Assume that all birds have feathers. We will show that all herons have feathers.

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Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
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Consider an arbitrary bird b. Since b is a bird, b has feathers.

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Consider an arbitrary bird b. Since b is a bird, b has feathers. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
    - Here, we ***assumed*** all birds have feathers.
    - Here, we ***proved*** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$\underbrace{(\forall b.\ (\mathit{Bird}(b) \to \mathit{Feathers}(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (\mathit{Heron}(h) \to \mathit{Feathers}(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- If we ***assume*** the statement

  $$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, have feathers.

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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
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$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{f is an involution.}} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{f is surjective.}}$$

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Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

We've said that we need to prove this statement. How do we do that?

What do you do to prove
$\forall b \in A$. [something]?

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To ***prove*** that
this is true…

Have the reader pick an arbitrary x. Then prove A is true for that choice of x.

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There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

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Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

Prove this.

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Prove this.

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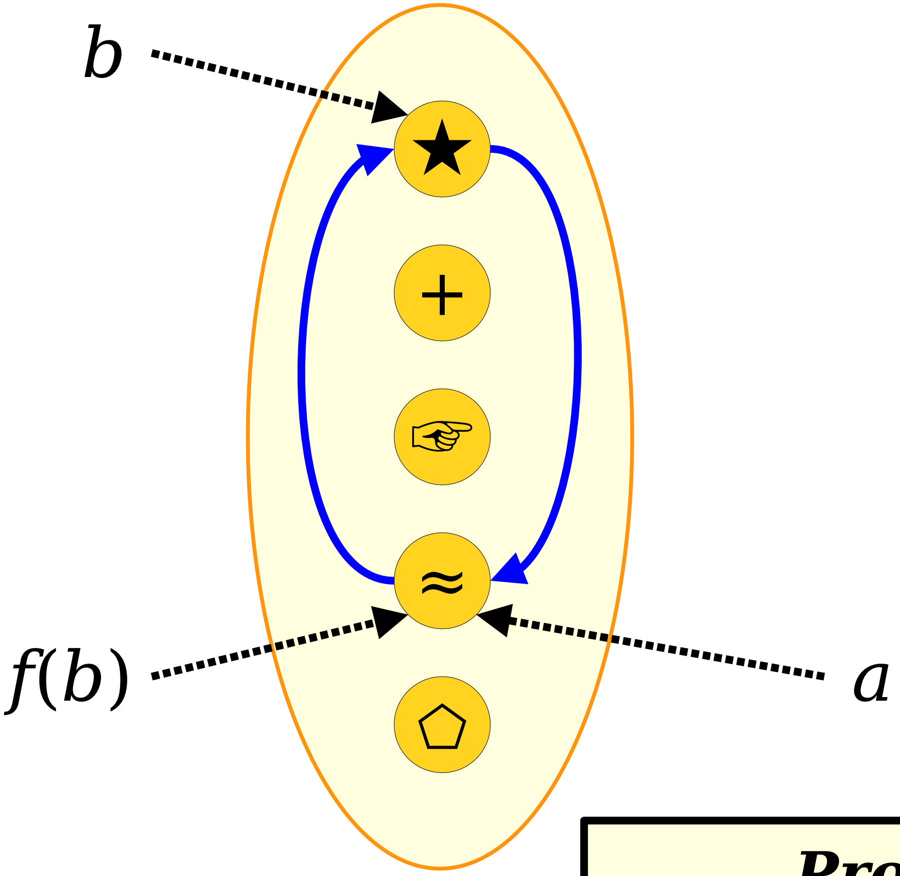
Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.



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Proof: Pick any involution $f : A \to A$. We will prove that f is surjective.

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Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

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The Two-Column Proof Organizer

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming

$f : A \to A$ is an involution.

$$\forall z \in A.\ f(f(z)) = z.$$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

What We Need to Prove

f is injective.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$$
$$a_1 = a_2$$
$$)$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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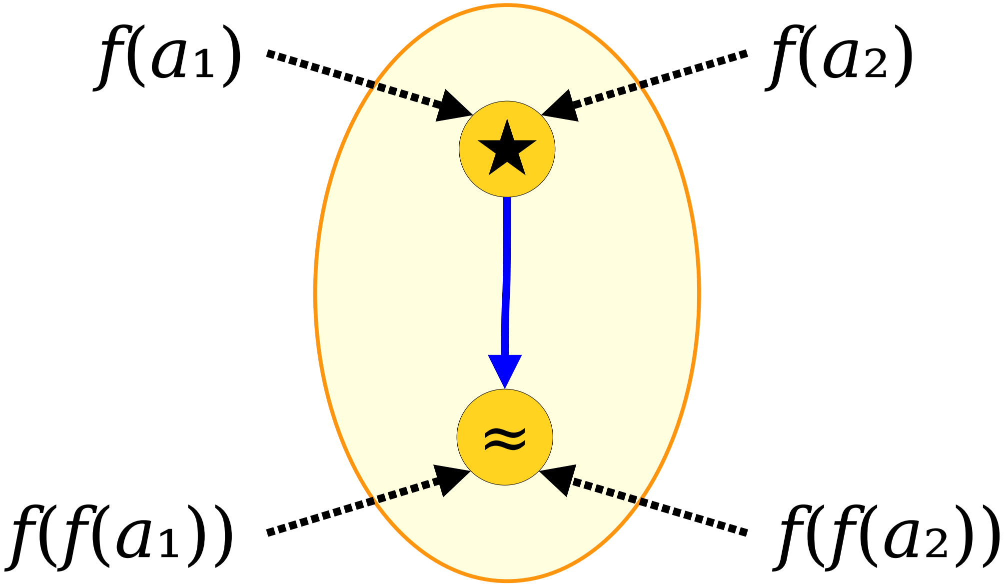

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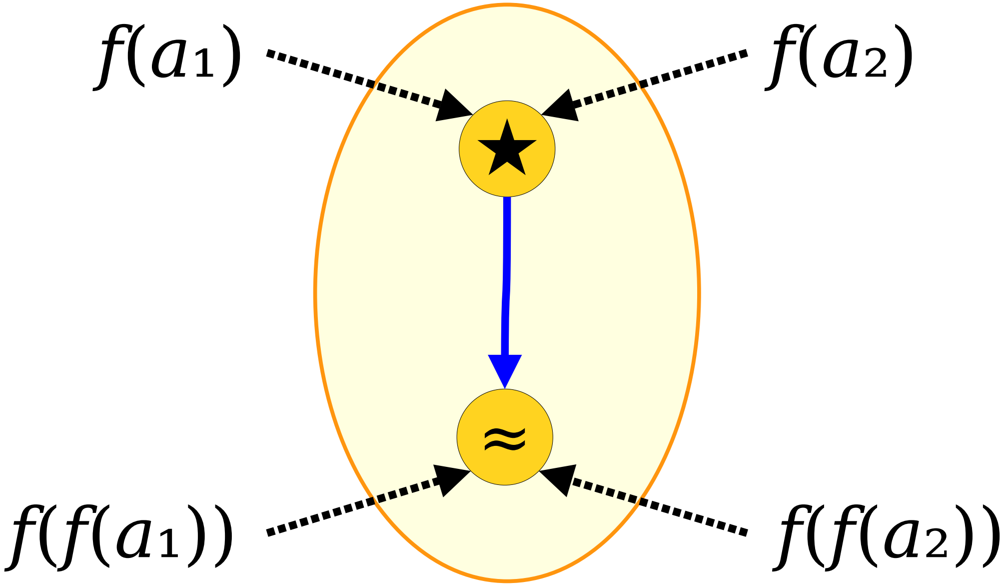

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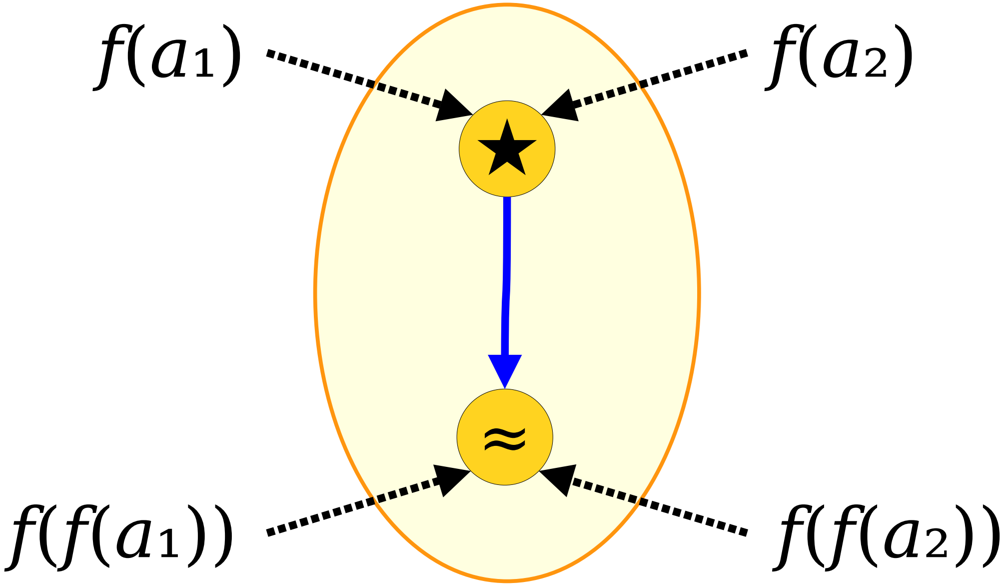

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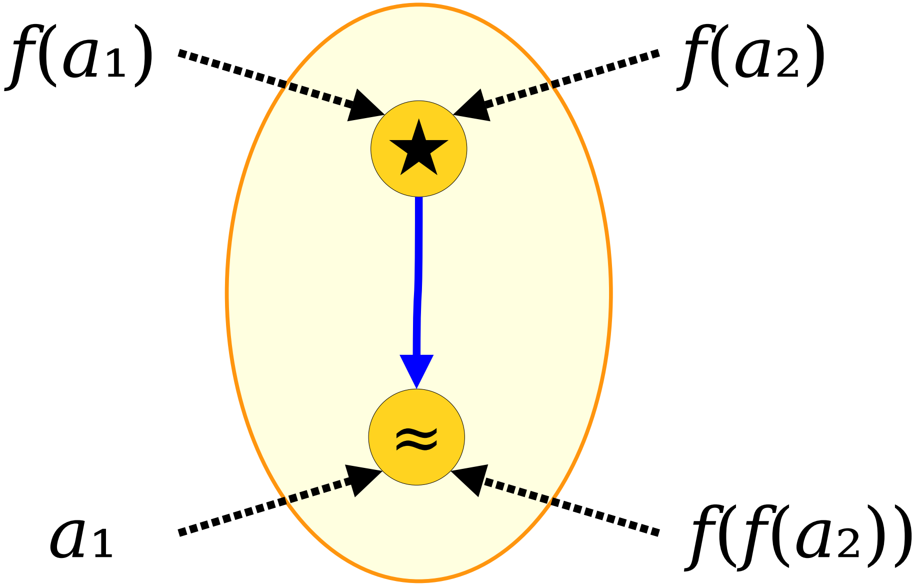

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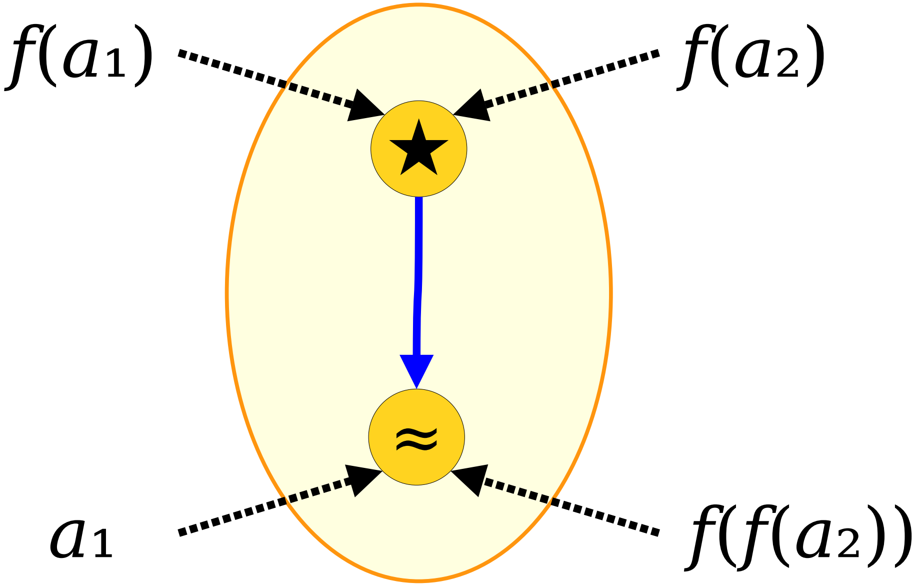

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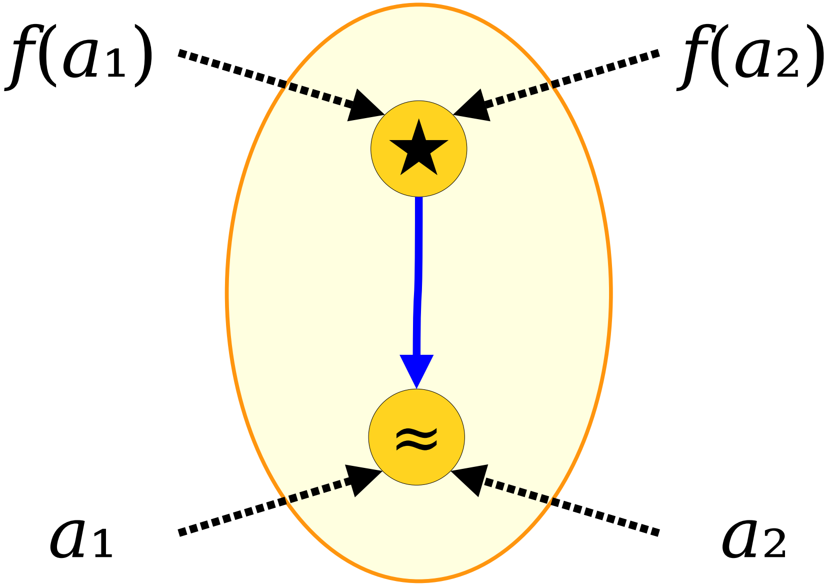

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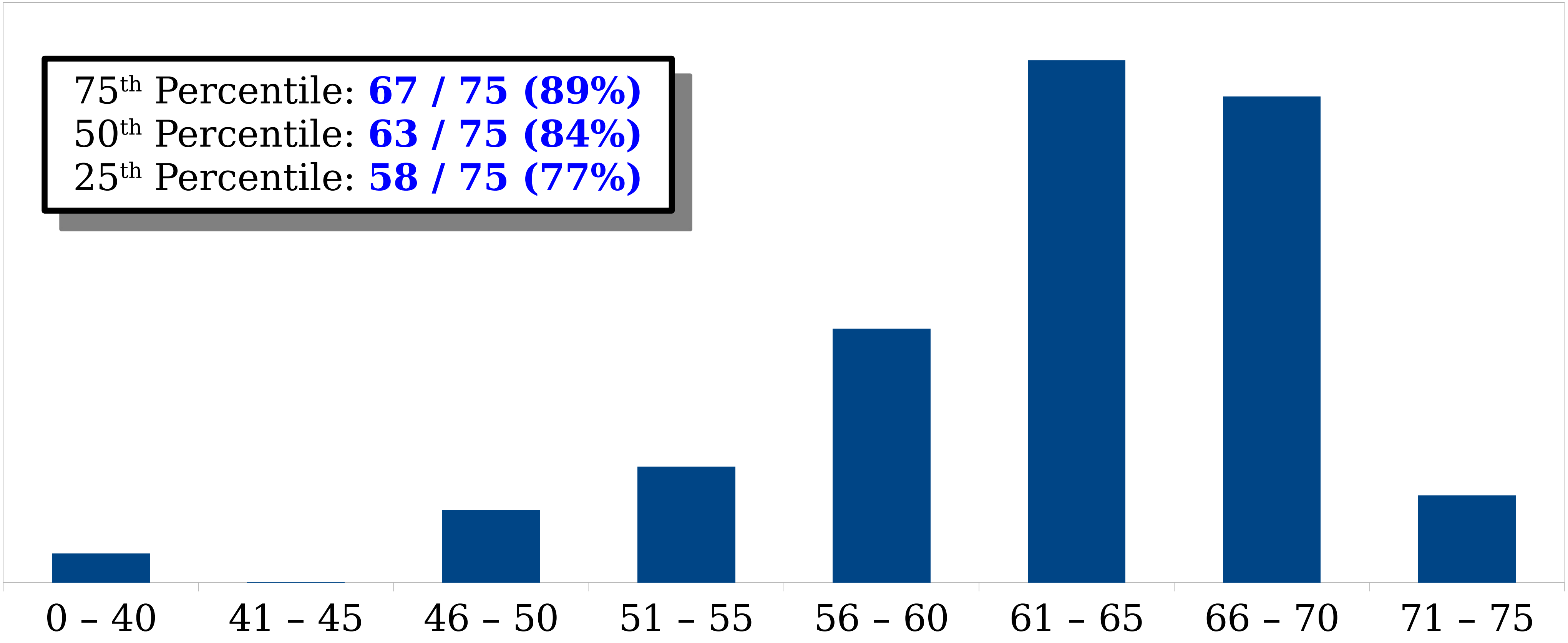
This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Time-Out for Announcements!

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set One Graded

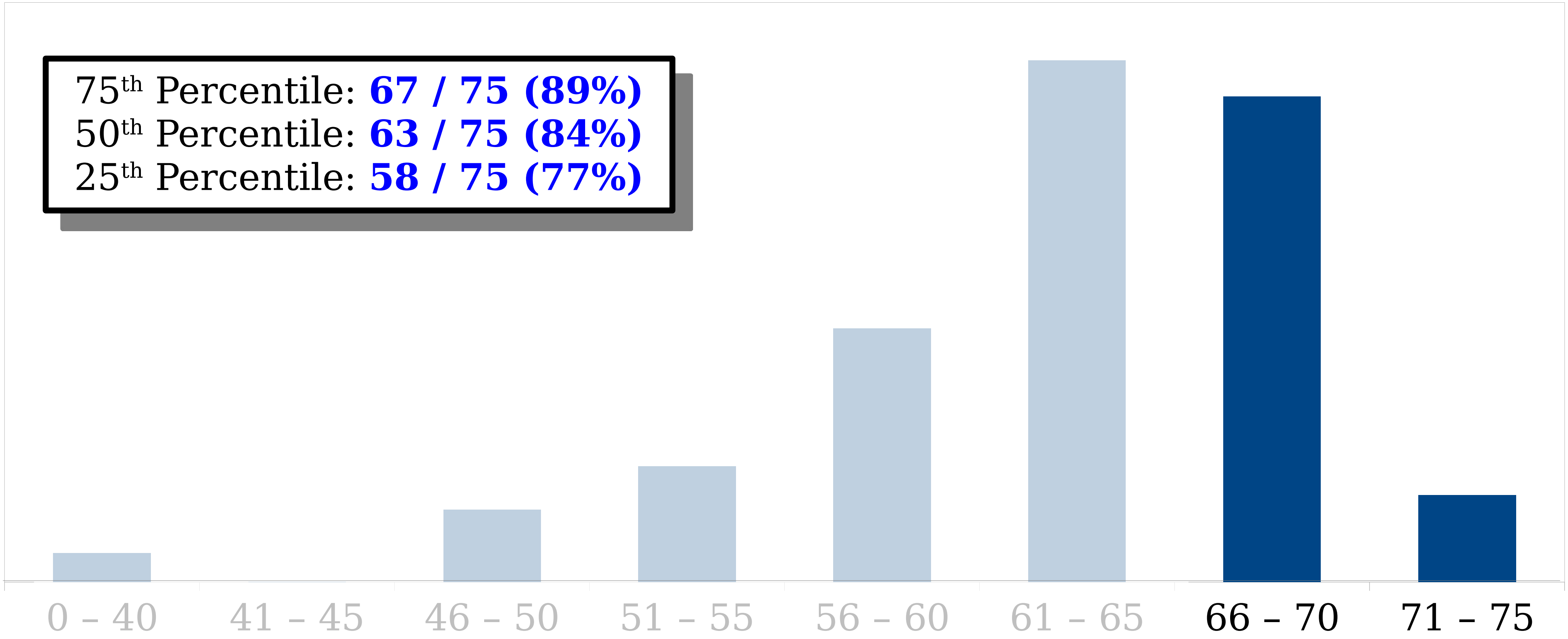


75th Percentile: **67 / 75 (89%)**
50th Percentile: **63 / 75 (84%)**
25th Percentile: **58 / 75 (77%)**

Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

Problem Set One Graded



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"Great job! Look over your feedback for some tips on how to tweak things for next time."

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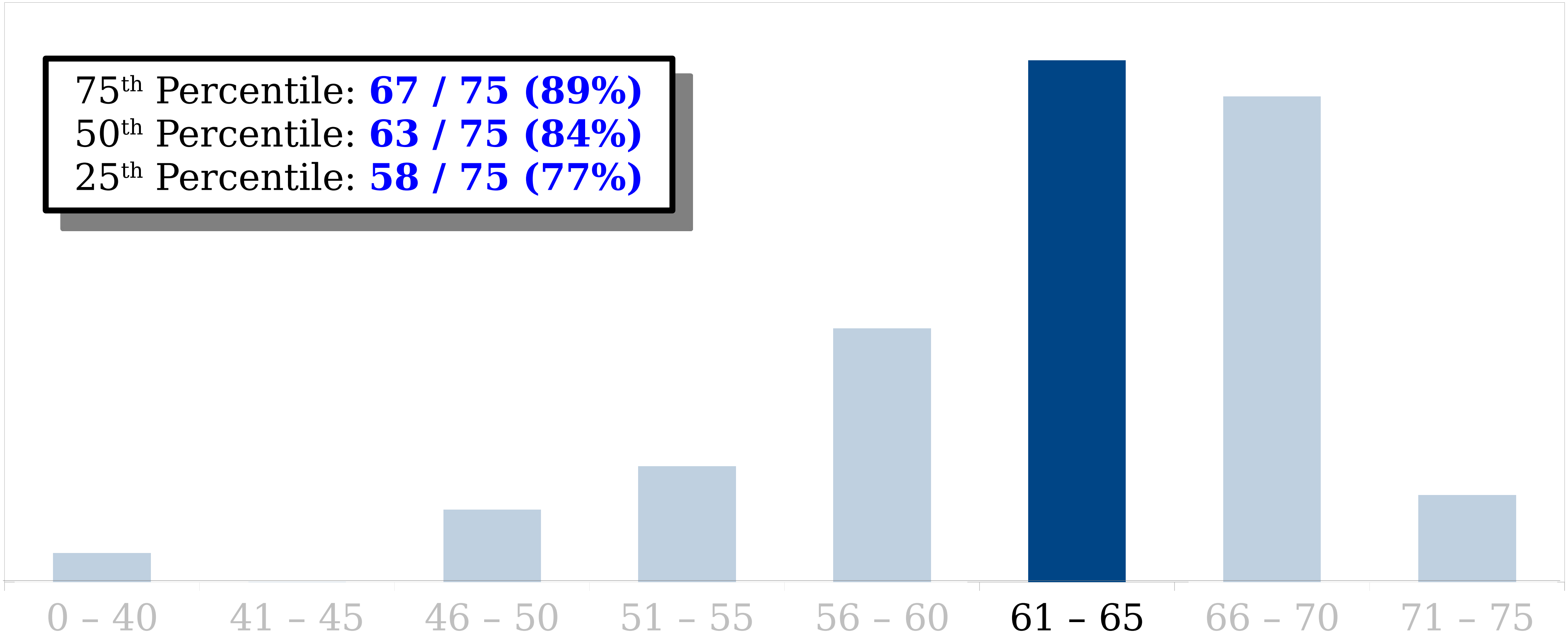

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"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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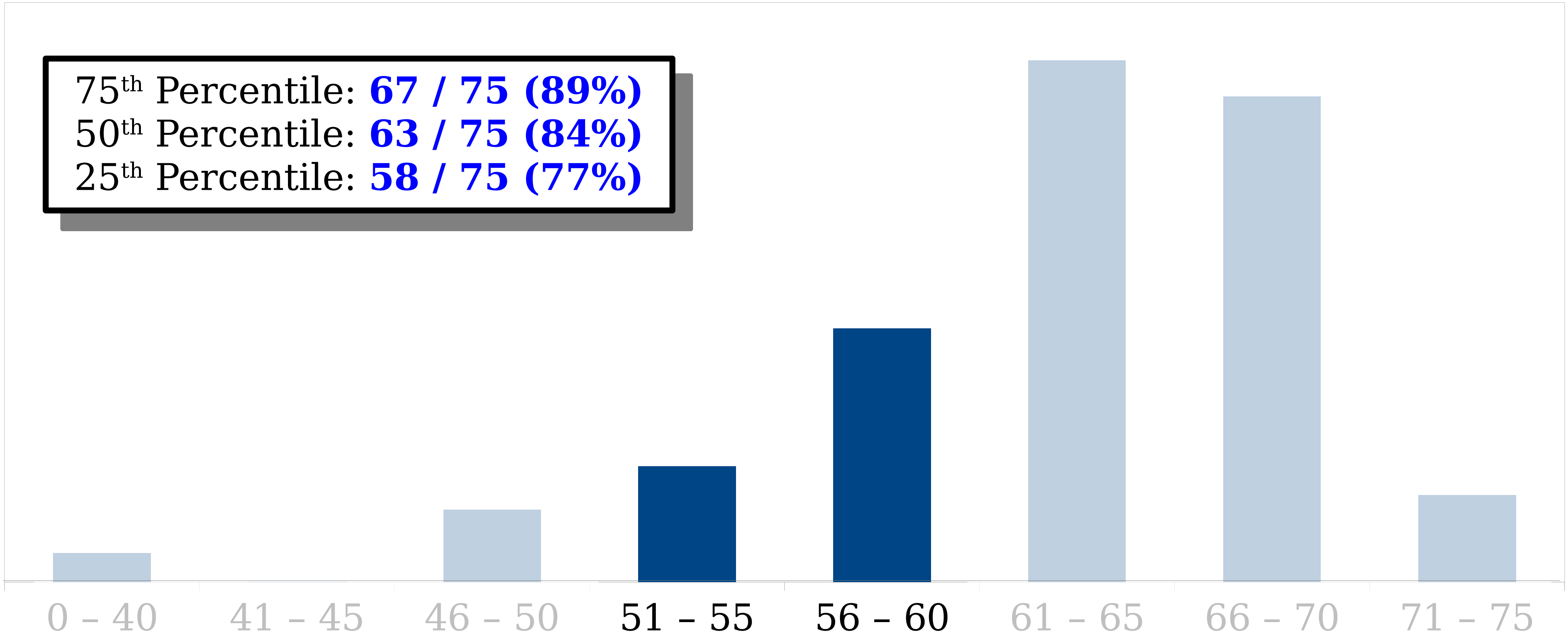
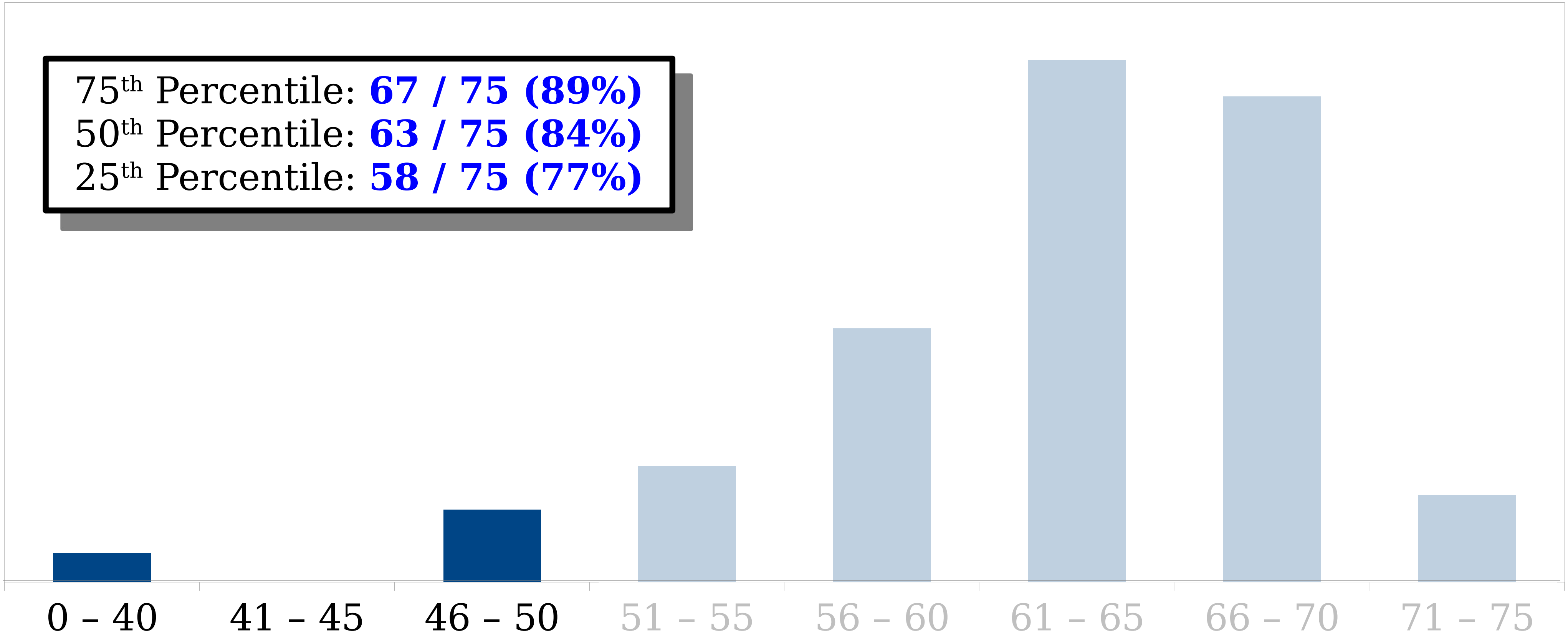


"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded



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"Looks like something hasn't quite clicked yet.
Get in touch with us and stop by office hours
to get some extra feedback and advice.
Don't get discouraged – you can do this!"

What Not to Think

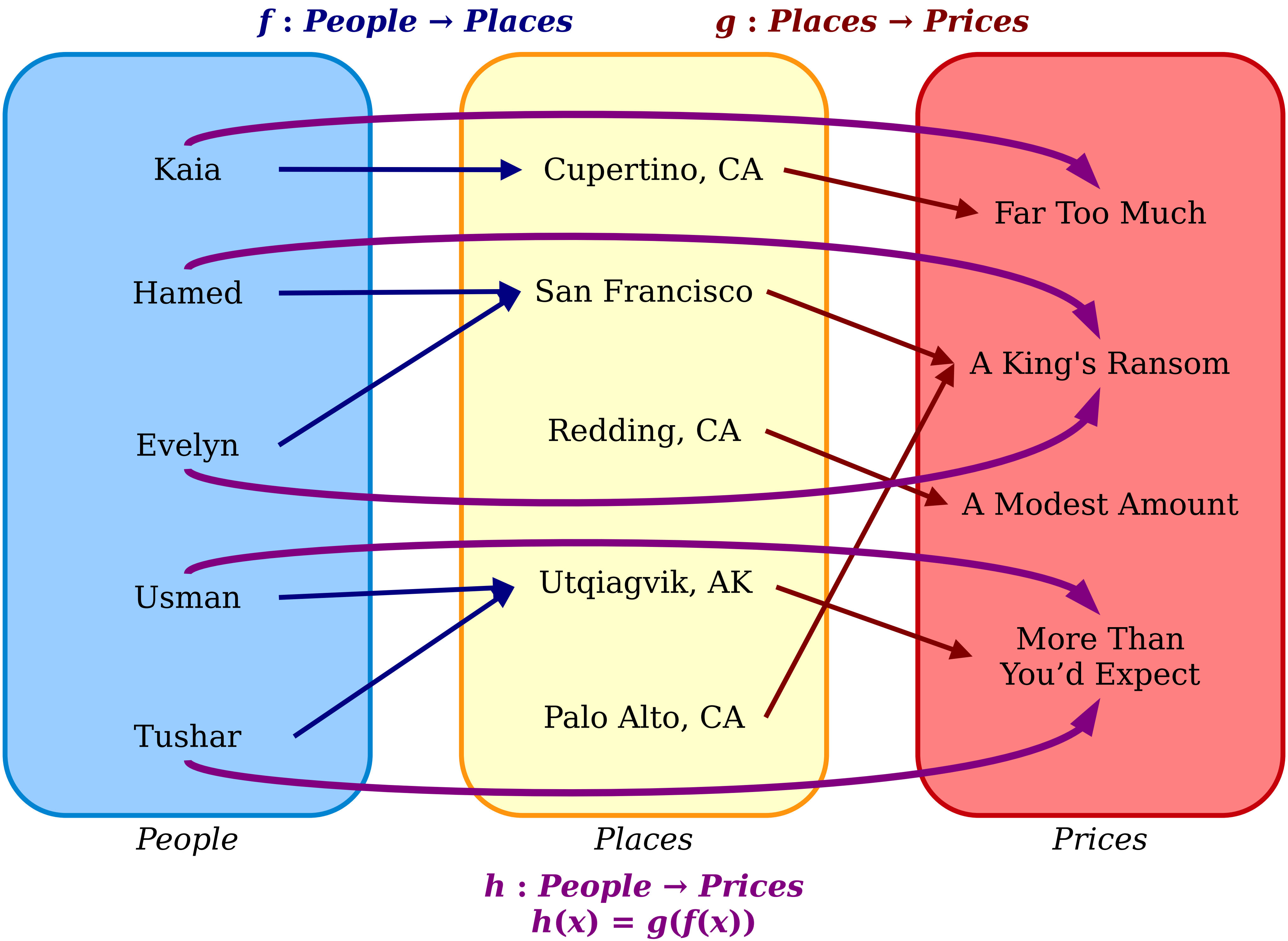
- "Well, I guess I'm just not good at math."
  - For most of you, this is your first time doing proof-based math.
  - It is ***totally normal*** when learning any new skill to have areas where you need to improve. And we cover a ton of material here!
  - You will improve over the quarter. Hang in there!
- "I got a good score, so I don't need to review anything."
  - Check your feedback. Make sure you didn't miss an important detail.
  - We let you work in pairs. Be honest with yourself – did you lean too much on your partner? Could you have done the work unassisted?
  - We provide lots of office hours. Be honest with yourself – did you get too much help from the TAs?
- You will need to be able to solve problems like these solo on the exams. ***Put in the time now to patch up any gaps in your understand***

Essential Action Items

- ***Review your feedback.***
  - Don’t just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
  - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
  - (Especially for Q8, Q10) Look at our solutions and see if there’s any neat lessons you can draw from them.
- ***Come to us with questions.***
  - Anything you’re not sure about? That’s what we’re here for! Come to office hours, ask questions on EdStem, etc.

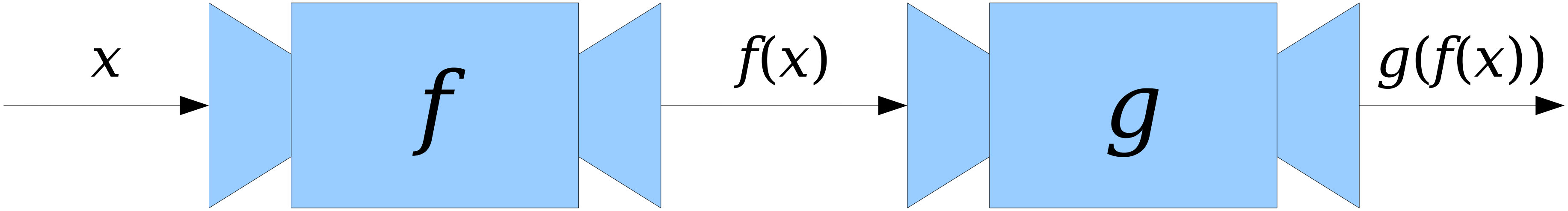
Back to CS103!

Function Composition


f : People → Places
g : Places → Prices
Kaia
Hamed
Evelyn
Usman
Tushar
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than
You’d Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is g's codomain.
  - Even though composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.
- Composition is ***associative***: $(f \circ g) \circ h = f \circ (g \circ h)$. (Prove this!)
- Composition is not necessarily commutative: $f \circ g$ is not necessarily the same as $g \circ f$. (Prove this!)

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to \\ f(x) \neq f(y) \\ )$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to \\ g(x) \neq g(y) \\ )$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to \\ (g \circ f)(a_1) \neq (g \circ f)(a_2) \\ )$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

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Let's write this out separately and simplify things a bit.

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$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

$g(f(a_1)) \neq g(f(a_2))$

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to \\ f(x) \neq f(y) \\ )$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to \\ g(x) \neq g(y) \\ )$$

$a_1 \in A$ is arbitrarily-chosen.

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What We Need to Prove

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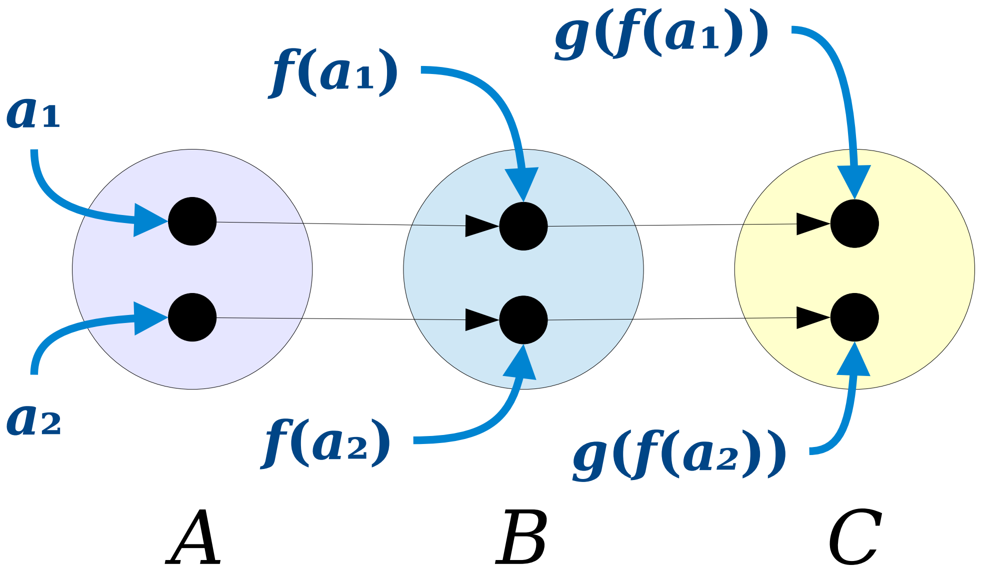
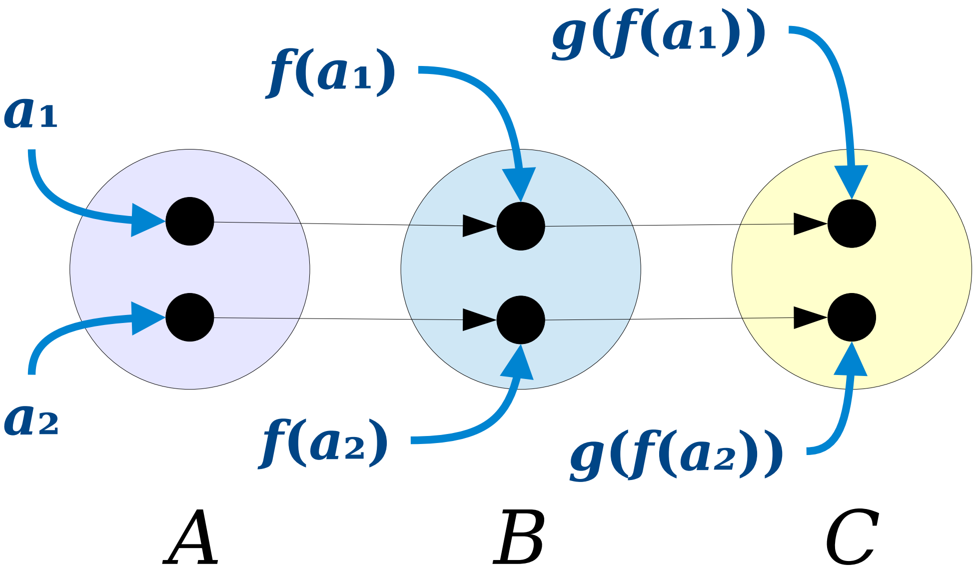
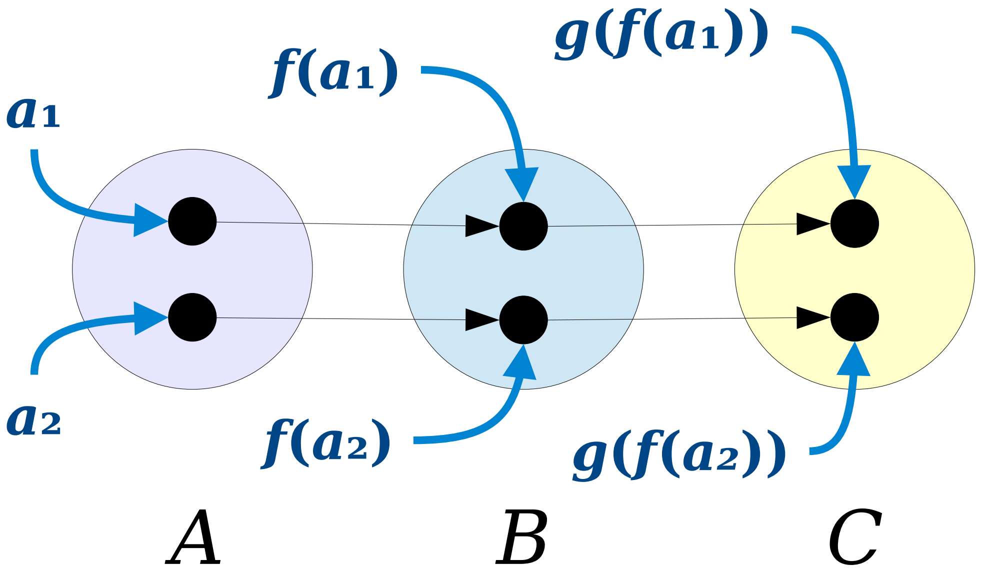


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



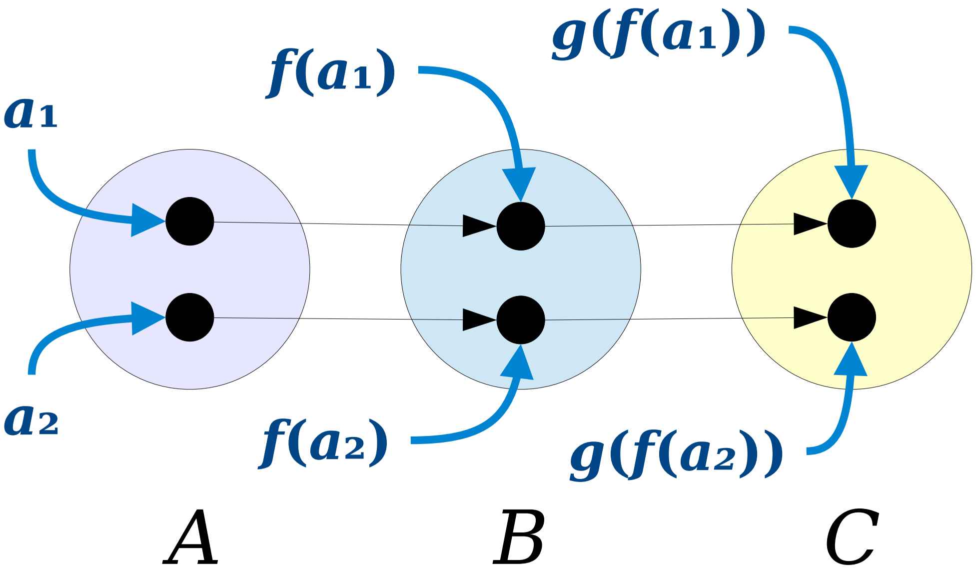
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof:



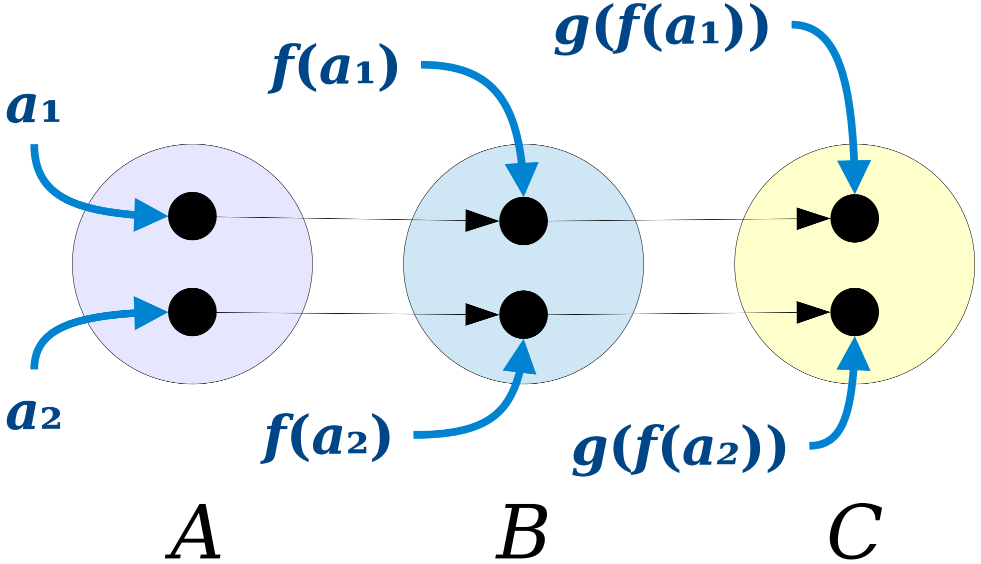
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



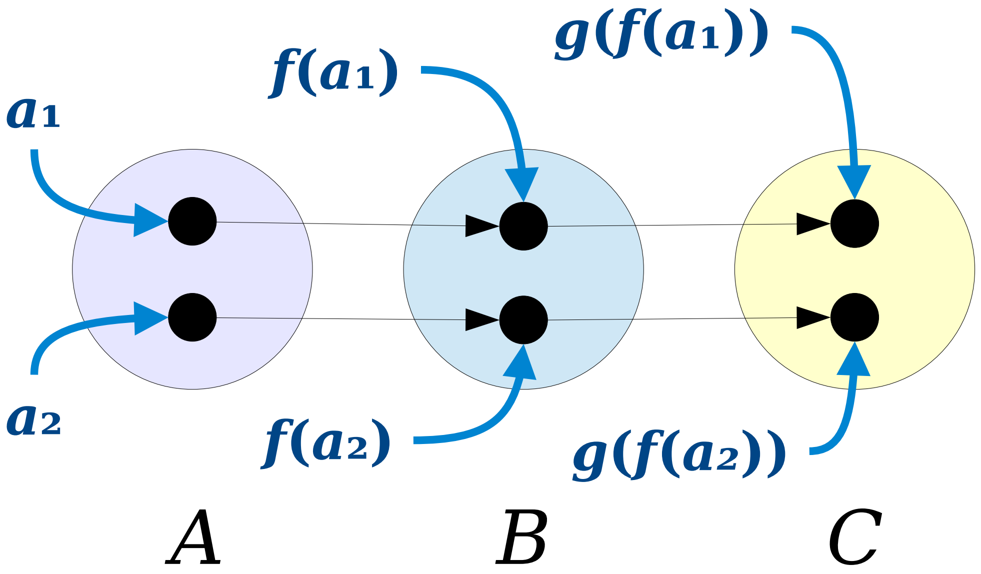
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$.



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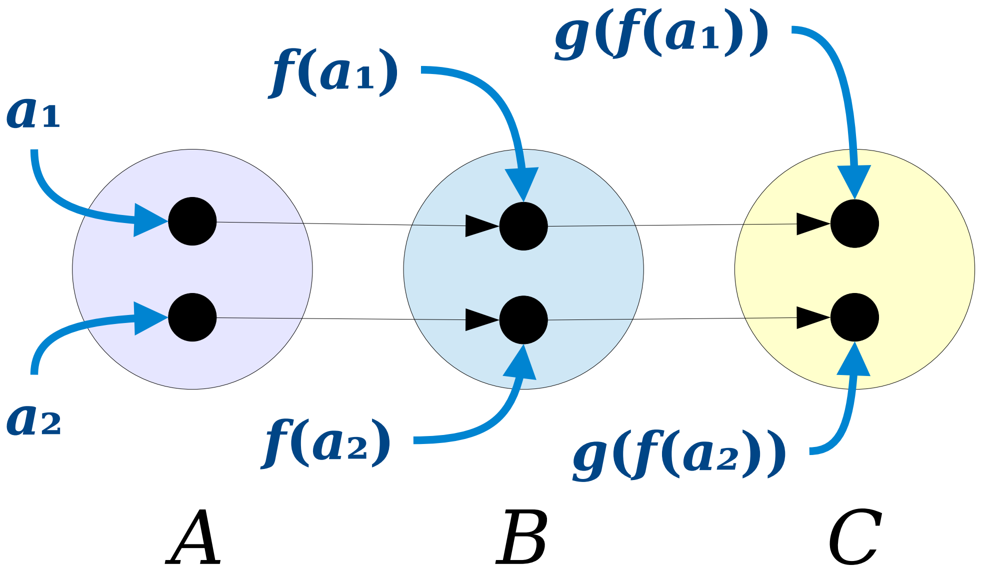
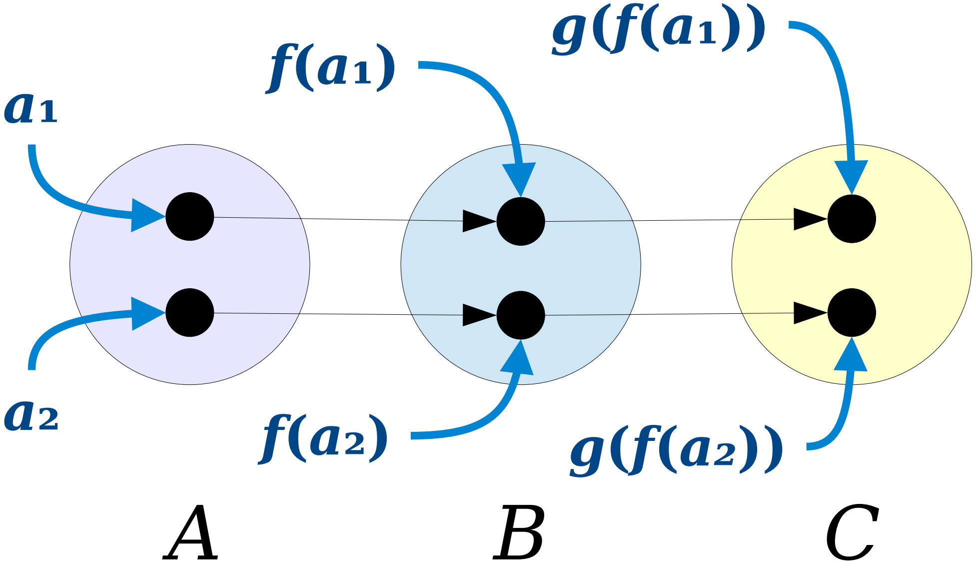


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.



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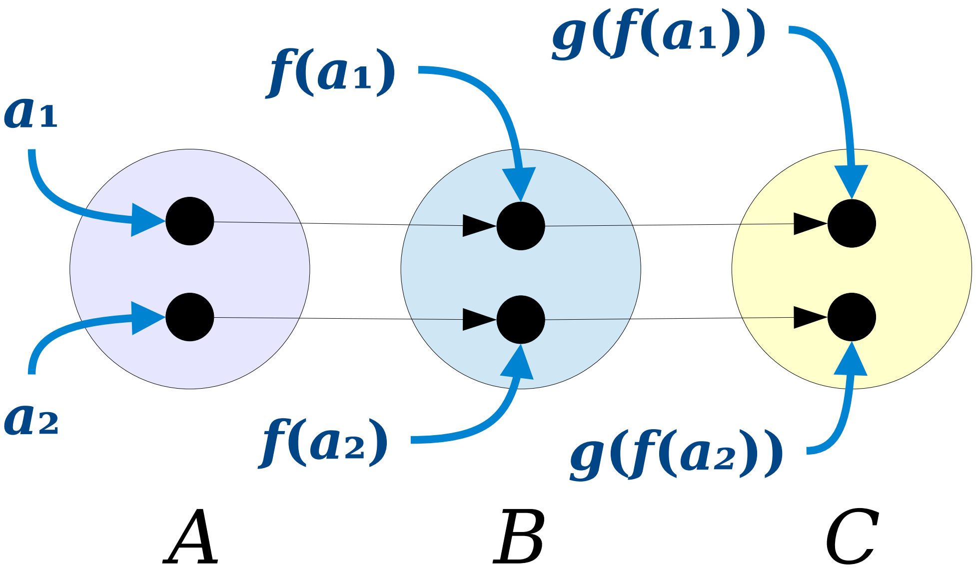
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



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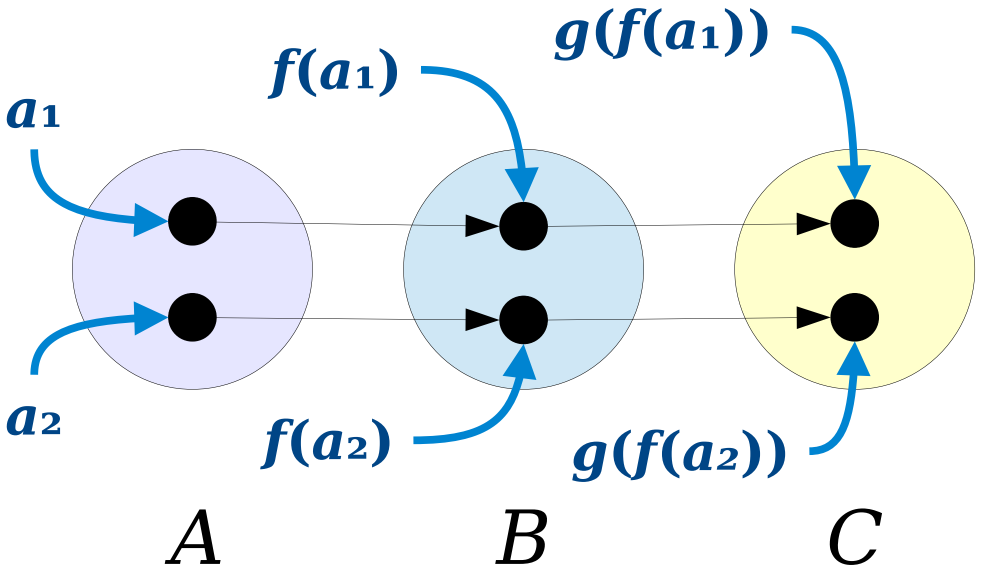
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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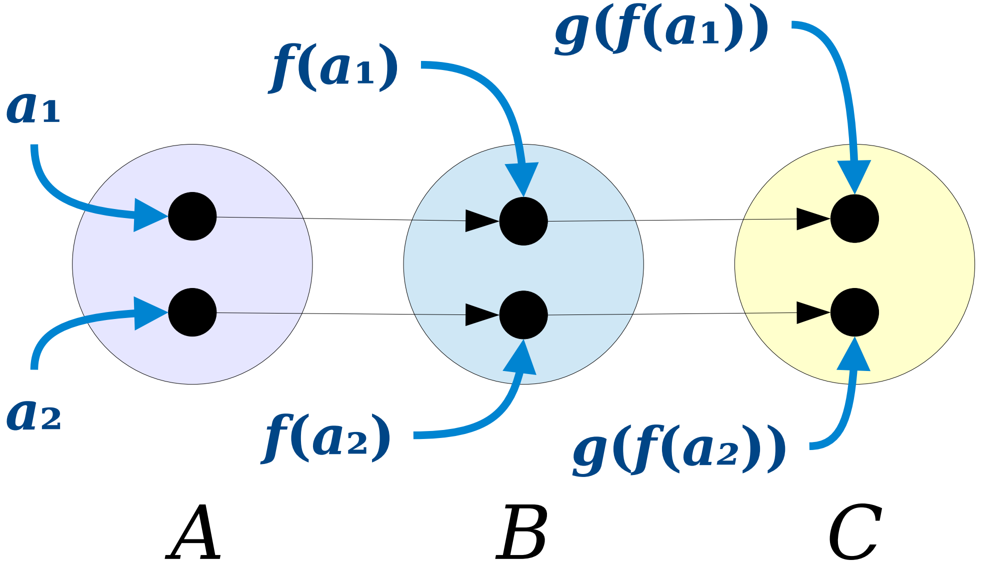
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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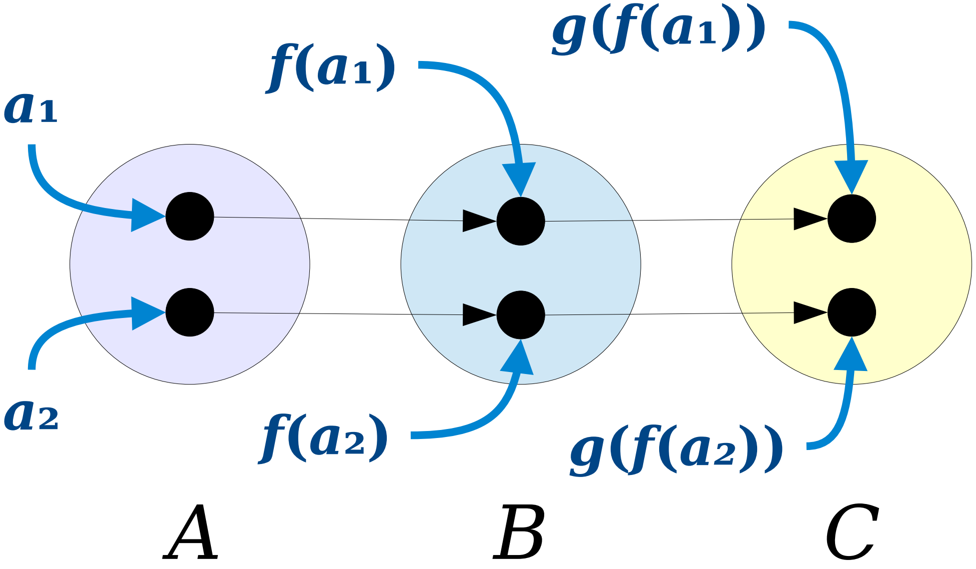
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



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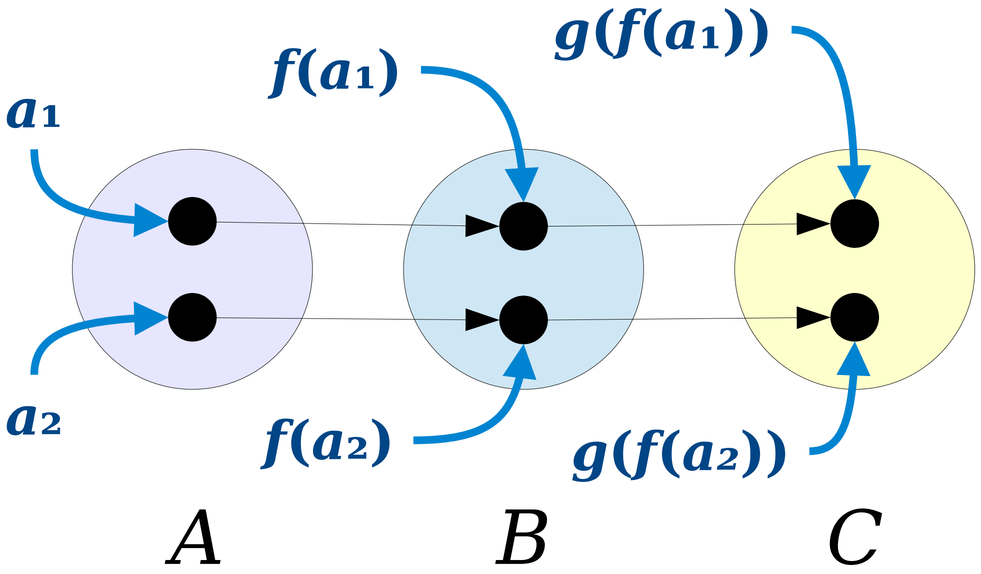
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Proofs on Sets***
  - How to rigorously establish set-theoretic results.

Appendix: Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections.

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What does it mean for $g \circ f : A \to C$ to be surjective?

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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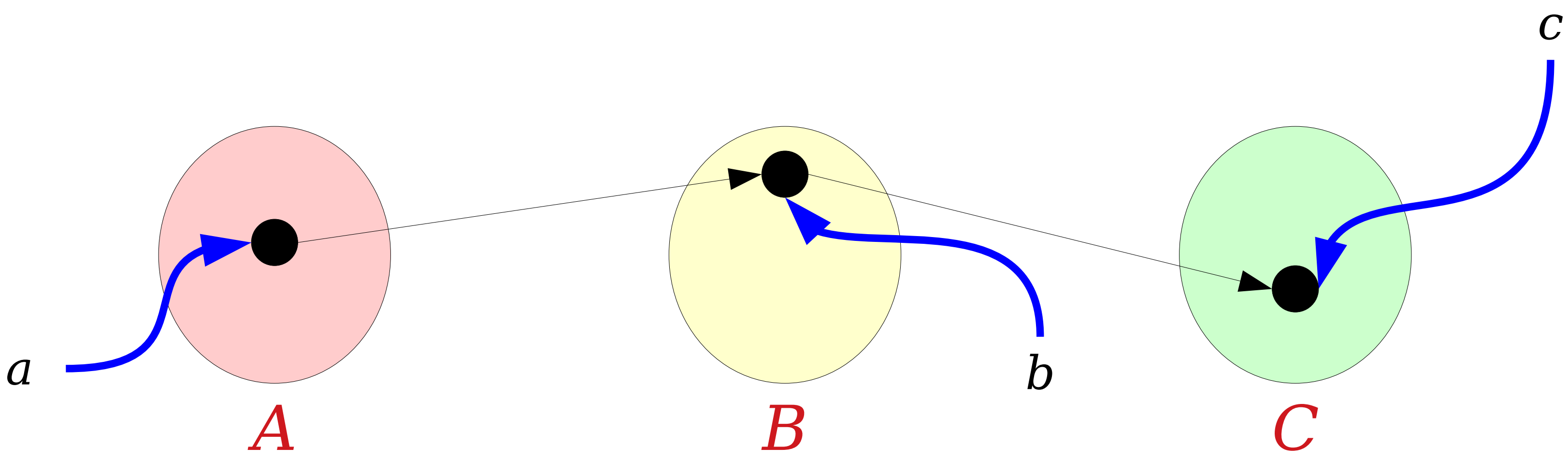

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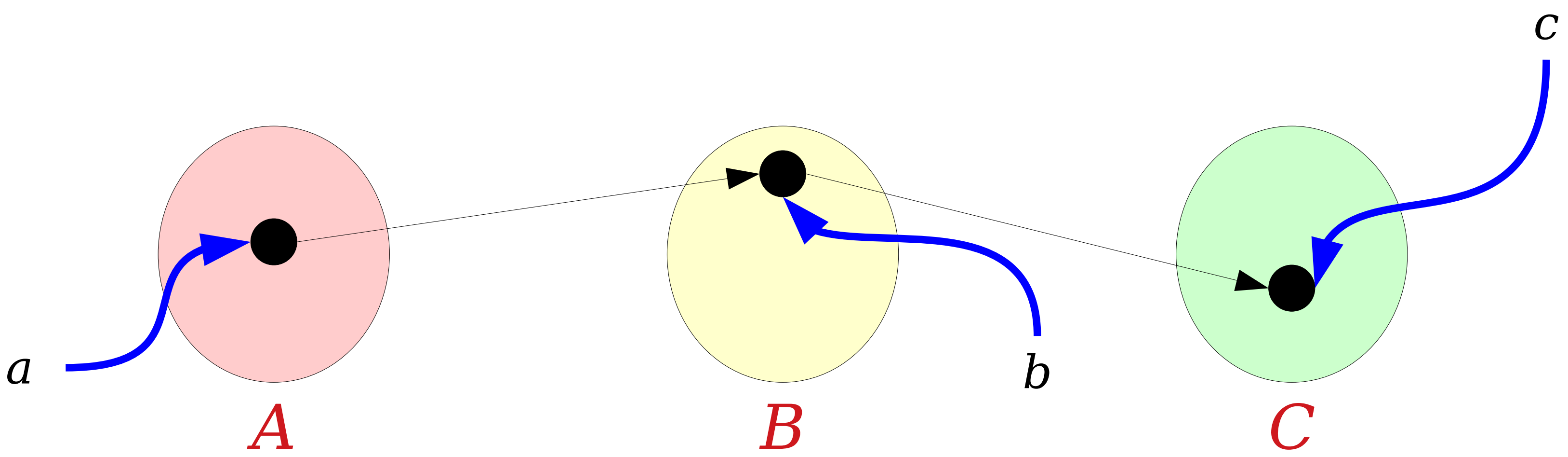
Consider any $c \in C$.



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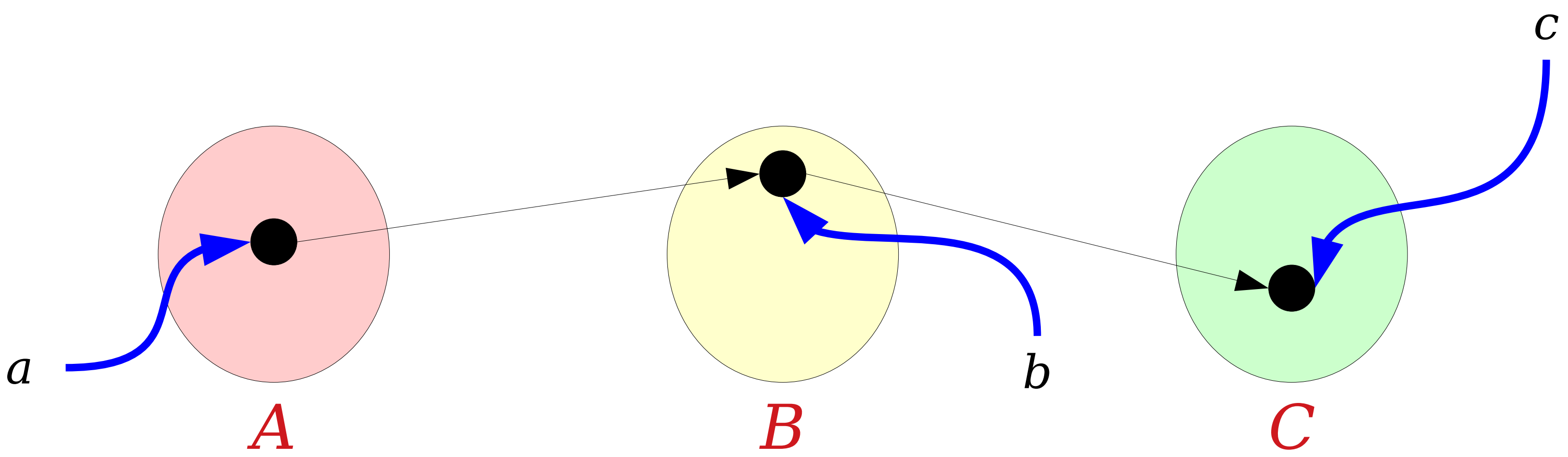
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$.



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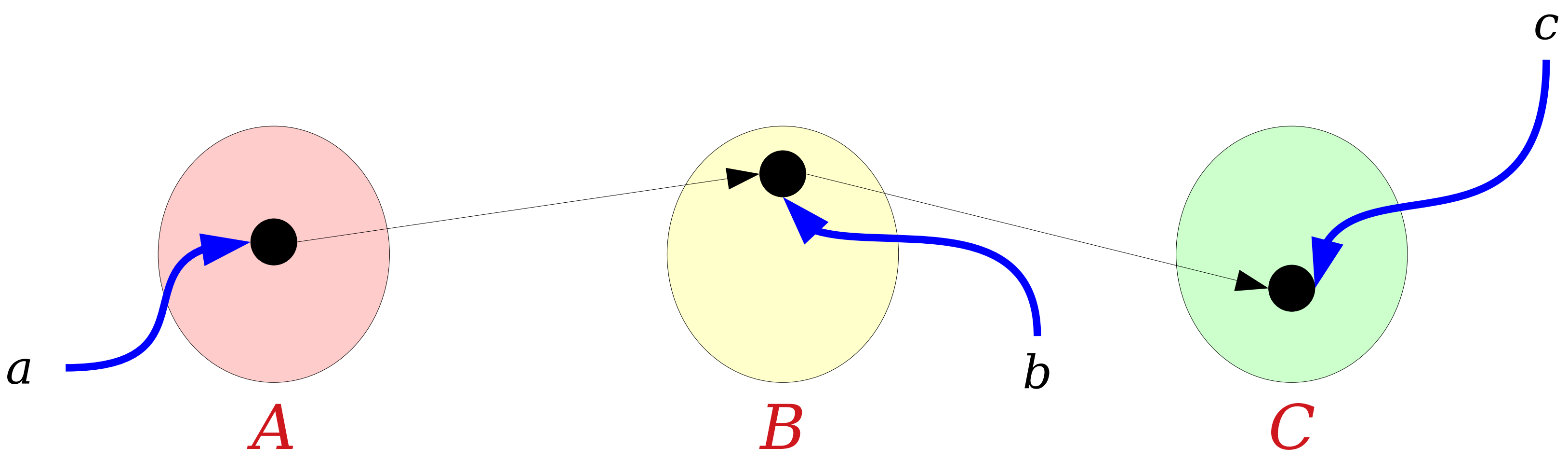
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



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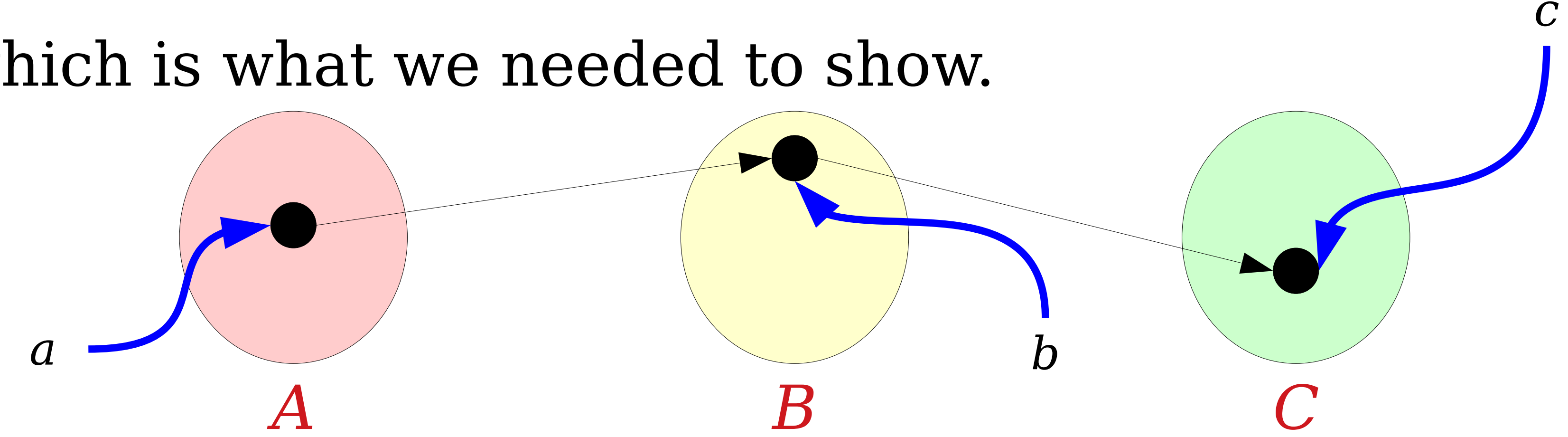
$$g(f(a)) = g(b) = c,$$

which is what we needed to show.



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